North Sydney Boys High School

100 Years 1912 ~ 2012

MATHEMATICS (EXTENSION 1)

2012 HSC Course Assessment Task 1

November 25, 2011

General instructions

- Working time – 50 minutes.
- Commence each new question on a new page. Write on both sides of the paper.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

Class (please ✔)

- ◯ 12M3C – Mr Lowe
- ◯ 12M3D – Mr Berry
- ◯ 12M3E – Mr Lam
- ◯ 12M4A – Mr Fletcher
- ◯ 12M4B – Mr Ireland
- ◯ 12M4C – Mr Weiss

NAME: ....

 # BOOKLETS USED: ....

Marker's use only.

QUESTION	1	2	3	4	5	Total	%
MARKS	$\overline{9}$	$\overline{9}$	$\overline{7}$	$\overline{9}$	$\overline{15}$	$\overline{48}$	

Question 1 (9 Marks) Commence a NEW page. **Marks**

(a) Find the vertex and focus of $(x-4)^2 = 2y - 6$. **3**

(b) Find the equation of the locus of a point $P(x, y)$ which moves such that its distance from the line $x = 8$ is twice its distance from the point $(2, 0)$. **3**

(c) A parabola has its focus at $(2, -4)$ and its directrix is the x axis.

i. Determine its vertex and write down the equation of the parabola. **2**

ii. What is the length of the latus rectum of the parabola? **1**

Question 2 (9 Marks) Commence a NEW page. **Marks**

(a) If α, β and γ are roots of the equation $2x^3 - 5x + 3 = 0$, find the value of

i. $\alpha\beta + \alpha\gamma + \beta\gamma$ **1**

ii. $\alpha\beta\gamma$ **1**

iii. $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **2**

iv. $\dfrac{1}{\alpha^2 + \beta^2 + \gamma^2}$ **2**

(b) The polynomial $x^3 + 2x^2 + ax + b$ has a factor of $(x + 2)$ and when divided by $(x - 2)$ leaves a remainder of 12. Find the value of a and b. **3**

Question 3 (7 Marks) Commence a NEW page. **Marks**

(a) Write down the equation of the chord of contact of the parabola $x^2 = 8y$ from the external point $P(4, -6)$. Do *not* derive the equation. **2**

(b) Given the quadratic equation $x^2 + (m - 6)x - 8m = 0$

i. Find the value of m if the roots are reciprocal of one another. **2**

ii. For what values of m does the quadratic $x^2 + (m - 6)x - 8m = 0$ have real roots? **3**

Question 4 (9 Marks) Commence a NEW page. **Marks**

(a) i. Show that $(x - 1)$ is a factor of $P(x) = x^3 - 6x^2 + 11x - 6$. **1**

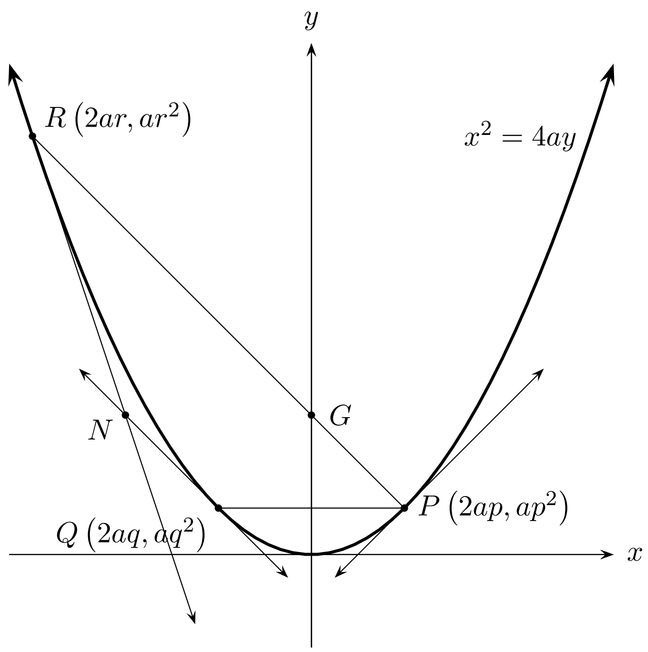
ii. Express $P(x)$ as a product of its factors. **3**

iii. Without using calculus, solve the inequality $x^4 - 6x^3 + 11x^2 - 6x \leq 0$. *Hint:* a sketch may be useful. **2**

(b) Solve $\left(x + x^{-1}\right)^2 - 6\left(x + x^{-1}\right) + 8 = 0$. **3**

Question 5 (15 Marks) Commence a NEW page. **Marks**

$P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ are points on the parabola $x^2 = 4ay$.
PQ is a chord perpendicular to the axis of symmetry of the parabola.



(a) Derive the equation of the tangent to the parabola at point P. **2**

(b) Prove that the equation of the normal at P is $x + py = 2ap + ap^3$ **2**

The normal at P cuts the axis of symmetry at G and the parabola again at R.

(c) Find the coordinates of G. **2**

(d) Find the locus of point G as P moves along the parabola. **1**

(e) The tangents at Q and R meet at N. Show that N has coordinates **3**

$$\Big(a(q + r), aqr\Big)$$

(f) Prove that $r = -p - \dfrac{2}{p}$. **2**

(g) Prove that NG is perpendicular to the axis of symmetry. **2**

End of paper.

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Suggested Solutions

Question 1 (Fletcher)

(a) (3 marks)

✓ [1] for correct focal length a.

✓ [1] for vertex.

✓ [1] for focus.

$$(x-4)^2 = 2y - 6$$
$$(x-4)^2 = 2(y-3) = 4 \times \frac{1}{2}(y-3)$$
$$\therefore a = \tfrac{1}{2} \quad V(4,3) \quad S\left(4, \tfrac{7}{2}\right)$$

(b) (3 marks)

✓ [1] for correct distance formulae.

✓ [1] for correct equation.

✓ [1] for correct locus.

Let the point $P(x, y)$ be the variable point that moves.

- The distance from P to $x = 8$ is
$$d_1 = \sqrt{(x-8)^2}$$
- The distance from P to $(2, 0)$ is
$$d_2 = \sqrt{(x-2)^2 + y^2}$$

As P moves s.t. the distance from $(2, 0)$ is twice that to $x = 8$,

$$\sqrt{(x-8)^2} = 2\sqrt{(x-2)^2 + y^2}$$
$$(x-8)^2 = 4\left(x^2 - 4x + 4 + y^2\right)^2$$
$$\therefore \underset{-x^2}{x^2} - \cancel{16x} + \underset{-16}{64} = \underset{-x^2}{4x^2} - \cancel{16x} + \underset{-16}{16} + 4y^2$$
$$3x^2 + y^2 = 48$$

(c) i. (2 marks)

✓ [1] for vertex.

✓ [1] for equation.

$$V(2, -2)$$
$$(x-2)^2 = -4(2)(y+2)$$
$$(x-2)^2 = -8(y+2)$$

ii. (1 mark)

Length of latus rectum $= 4a = 4 \times 2 = 8$

Question 2 (Ireland)

(a) $2x^3 + 0x^2 - 5x + 3 = 0$.
Note that $\alpha + \beta + \gamma = -\frac{b}{a} = 0$.

i. (1 mark)

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a} = -\frac{5}{2}$$

ii. (1 mark)

$$\alpha\beta\gamma = -\frac{d}{a} = -\frac{3}{2}$$

iii. (2 marks)

✓ [1] for correct expression.

✓ [1] for final answer.

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma} = \frac{-\frac{5}{2}}{-\frac{3}{2}} = \frac{5}{3}$$

iv. (2 marks)

✓ [1] for correct expression.

✓ [1] for final answer.

$$\frac{1}{\alpha^2 + \beta^2 + \gamma^2}$$
$$= \frac{1}{(\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)}$$
$$= \frac{1}{0 - 2\left(-\frac{5}{2}\right)} = \frac{1}{5}$$

(b) ✓ [1] for two equations in a and b.

✓ [1] each per correct value of a and b.

As $(x+2)$ is a factor of $P(x) = x^3 + 2x - ax + b$,

$$P(-2) = (-2)^3 + 2(-2)^2 - 2a + b = 0$$
$$-8 + 8 - 2a + b = 0$$
$$\therefore b = 2a \quad (1)$$

By the remainder theorem,

$$P(2) = (2)^3 + 2(2)^2 + 2a + b = 12$$
$$2a + b = -4 \quad (2)$$

Substitute (1) to (2),

$$2a + b = b + b = -4$$
$$b = -2$$
$$\therefore a = -1$$

Question 3 (Berry)

(a) (2 marks)

✓ [1] for correct equation of the chord $xx_0 = 2a(y + y_0)$.

✓ [1] for final answer.

$$\begin{aligned} x^2 &= 8y = 4 \times 2y \\ \therefore a &= 2 \end{aligned}$$

The equation of the chord of contact is

$$xx_0 = 2a(y + y_0)$$

From $(4, -6)$ with focal length $a = 2$,

$$\begin{aligned} 4x &= 4(y - 6) \\ x &= y - 6 \\ \therefore y &= x + 6 \end{aligned}$$

(b) i. (2 marks)

✓ [1] for $\alpha\beta = \frac{c}{a} = 1$.

✓ [1] for correct value of m.

$$x^2 + (m - 6)x - 8m = 0$$

Roots are α & $\frac{1}{\alpha}$.

$$\begin{aligned} \alpha \times \frac{1}{\alpha} &= \frac{c}{a} \\ \frac{-8m}{1} &= 1 \\ \therefore m &= -\frac{1}{8} \end{aligned}$$

ii. (3 marks)

✓ [1] for $\Delta = m^2 + 20m + 36$.

✓ [1] for identifying that $\Delta \geq 0$.

✓ [1] for final answer.

$$\begin{aligned} \Delta &= b^2 - 4ac \\ &= (m - 6)^2 - 4(1)(-8m) \\ &= m^2 - 12m + 36 + 32m \\ &= m^2 + 20m + 36 \\ &= (m + 18)(m + 2) \end{aligned}$$

Real roots occur when $\Delta \geq 0$:

$$\begin{aligned} (m + 18)(m + 2) &\geq 0 \\ \therefore m \leq -18 \quad &\text{or} \quad m \geq -2 \end{aligned}$$

Question 4 (Lam)

(a) $P(x) = x^3 - 6x^2 + 11x - 6$

i. (1 mark)

$$P(1) = 1 - 6 + 11 - 6 = 0$$

By the factor theorem, $x - 1$ is a factor of $P(x)$.

ii. (3 marks)

✓ [3] for correctly factorising.

✓ [2] only for $(x - 1)(x - 6)(x + 1)$.

✓ [1] for one further error, depending on solution provided.

- By long division,

$$\begin{array}{r|l} & x^2 - 5x + 6 \\ \hline x - 1 & x^3 - 6x^2 + 11x - 6 \\ & -x^3 + x^2 \\ \hline & -5x^2 + 11x \\ & 5x^2 - 5x \\ \hline & 6x - 6 \\ & -6x + 6 \\ \hline & 0 \end{array}$$

$$\begin{aligned} \therefore P(x) &= (x - 1)(x^2 - 5x + 6) \\ &= (x - 1)(x - 2)(x - 3) \end{aligned}$$

- By guessing $x - 2$ is a factor:

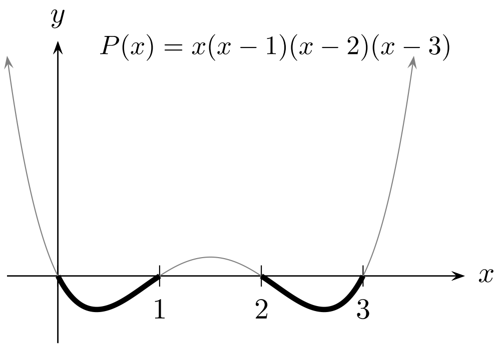
$$\begin{aligned} P(2) &= 2^3 - 6(2)^2 + 11(2) - 6 \\ &= 8 - 24 + 22 - 6 = 0 \end{aligned}$$

Similarly, use factor theorem again for $x - 3$ to show it is also a factor.

iii. (2 marks)

✓ [1] for correct method (usually correct graph).
If any other graph is (correctly) sketched and resulting in incorrect inequalities (i.e. $P(x) \leq 0$), only [1] is awarded. [0] for incorrect graph and incorrect inequalities (2 errors).

✓ [1] for evaluating $P(x) \leq 0$ correctly (i.e. correct inequality)



From the graph, $P(x) \leq 0$ when

$$0 \leq x \leq 1 \quad \text{or} \quad 2 \leq x \leq 3$$

(b) (3 marks)

- ✓ [1] for method which leads to correct solutions.
- ✓ [1] for $x = 1$.
- ✓ [1] for $x = 2 \pm \sqrt{3}$.
- ✓ [−1] for non exact values.

$$\left(x + \frac{1}{x}\right)^2 - 6\left(x + \frac{1}{x}\right) + 8 = 0$$

Let $m = \left(x + \frac{1}{x}\right)$,

$$\begin{aligned} m^2 - 6m + 8 &= 0 \\ (m-4)(m-2) &= 0 \\ \therefore m &= 2, 4 \\ \therefore \left(x + \frac{1}{x}\right) &= 2, 4 \end{aligned}$$

Evaluating $x + \frac{1}{x} = 2$,

$$\begin{aligned} \underbrace{x + \frac{1}{x}}_{\times x} &= \underset{\times x}{2} \\ x^2 + 1 &= 2x \\ x^2 - 2x + 1 &= 0 \\ (x-1)^2 &= 0 \\ \therefore x &= 1 \end{aligned}$$

Evaluating $x + \frac{1}{x} = 4$,

$$\begin{aligned} \underbrace{x + \frac{1}{x}}_{\times x} &= \underset{\times x}{4} \\ x^2 + 1 &= 4x \\ x^2 - 4x + 1 &= 0 \\ x &= \frac{4 \pm \sqrt{4^2 - 4(1)(1)}}{2} \\ &= \frac{4 \pm \sqrt{12}}{2} = \frac{\cancel{2}\left(2 \pm \sqrt{3}\right)}{\cancel{2}} \\ &= 2 \pm \sqrt{3} \end{aligned}$$

Question 5 (Lowe/Weiss)

(a) (2 marks)

- ✓ [1] for correct gradient.
- ✓ [1] for correct equation.

$$y = \frac{x^2}{4a}$$

$$\frac{dy}{dx} = \frac{2x}{4a} = \left.\frac{x}{2a}\right|_{x=2ap} = \frac{\cancel{4a}p}{\cancel{4a}} = p$$

Apply the point gradient formula to find the equation of the tangent,

$$\begin{aligned} \frac{y - ap^2}{x - 2ap} &= p \\ y - \underset{+ap^2}{ap^2} &= px - \underset{+ap^2}{2ap^2} \\ y &= px - ap^2 \end{aligned}$$

(b) (2 marks)

- ✓ [1] for correct gradient.
- ✓ [1] for correct equation.

$$m_\perp = -\frac{1}{p}$$

Apply the point gradient formula to find the equation of the normal,

$$\begin{aligned} \frac{y - ap^2}{x - 2ap} &= -\frac{1}{p} \\ y - ap^2 &= -\frac{1}{p}(x - 2ap) \\ py - ap^3 &= -x + 2ap \\ \therefore x + py &= 2ap + ap^3 \end{aligned}$$

(c) (2 marks)

✓ [1] for correct point G.

✓ [1] for correct locus equation.

At $x = 0$, the equation of the normal becomes

$$py = 2ap + ap^3$$
$$\therefore y = 2a + ap^2$$
$$\therefore G\left(0, 2a + ap^2\right)$$

(d) (1 mark)
The locus of G is $x = 0$ (y axis)

(e) (3 marks)

✓ [1] for evaluating simultaneous equations involving tangents at Q and R.

✓ [1] for finding the x value.

✓ [1] for finding the y value.

Equations of tangents at Q and R are

$$\begin{cases} y = qx - aq^2 & ① \\ y = rx - ar^2 & ② \end{cases}$$

Equating ① and ② to find the point of intersection N,

$$qx - aq^2 = rx - ar^2$$
$$qx - rx = aq^2 - ar^2$$
$$x(q - r) = a(q^2 - r^2) = a(q - r)(q + r)$$
$$\therefore x = a(q + r) \qquad ③$$

Substitute ③ to ① to find the y coordinate:

$$y = q(a)(q + r) - aq^2$$
$$= \cancel{aq^2} + aqr - \cancel{aq^2}$$
$$= aqr$$
$$\therefore N\Big(a(q + r), aqr\Big)$$

(f) (2 marks)

✓ [1] for $2ar - 2ap = ap^3 - ar^2p$.

✓ [1] for completely showing $r = -p - \frac{2}{p}$.

As $R\left(2ar, ar^2\right)$ lies on the normal from P, substitute its coordinates into the equation of the normal:

$$x + py\Big|_{\substack{x=2ar\\y=ar^2}} = 2ap + ap^3$$
$$2ar + apr^2 = 2ap + ap^3$$
$$2ar - 2ap = ap^3 - apr^2$$
$$2\cancel{a}(r - p) = \cancel{a}p\left(p^2 - r^2\right)$$
$$\underset{\div p}{-2}\cancel{(p - r)} = \underset{\div p}{p}\cancel{(p - r)}(p + r)$$
$$\frac{-2}{p} = p + r$$
$$\therefore r = -p - \frac{2}{p}$$

Alternatively, obtain quadratic in r and solve as quadratic equation.

$$\cancel{a}pr^2 + 2\cancel{a}r - 2\cancel{a}p - \cancel{a}p^3 = 0$$
$$a = p \qquad b = 2 \qquad c = -(2p + p^3)$$
$$r = \frac{-2 \pm \sqrt{(2)^2 + 4(p)(2p + p^3)}}{2p}$$
$$= \frac{-2 \pm \sqrt{4 + 8p^2 + 4p^4}}{2p}$$
$$= \frac{-2 \pm \sqrt{4\left(p^2 + 1\right)^2}}{2p}$$
$$= \frac{-2 \pm 2\left(p^2 + 1\right)}{2p} = \frac{-2 \pm 2p^2 + 2}{2p}$$

$$r = \frac{-2 + 2p^2 + 2}{2p} = \frac{2p^2}{2p} = p \qquad \Bigg| \qquad r = \frac{-2 - 2p^2 - 2}{2p} = \frac{-4 - 2p^2}{2p} = -p - \frac{2}{p}$$

As $r \neq p$, then $r = -p - \frac{2}{p}$ only.

(g) (2 marks)

✓ [1] noting $p = -q$ and substitutes this into coordinates of N.

✓ [1] for correct working.

Since PQ is $\perp$ to y axis, $\therefore p = -q$. Also, $r = -p - \frac{2}{p}$.

$$y = aqr = a(-p)\left(-p - \frac{2}{p}\right)$$
$$= ap\left(p + \frac{2}{p}\right)$$
$$= ap^2 + 2a$$
$$\therefore N\Big(a(q + r), 2a + ap^2\Big)$$

As $G\left(0, 2a + ap^2\right)$ and $N\left(a(q + r), 2a + ap^2\right)$, hence NG is perpendicular to the axis of the parabola.